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Convergence of Relaxation Schemes for Initial Boundary Value Problems for Conservation Laws

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Abstract—We consider a scalar conservation law with stiff source term in the quarter plan. This equation is relaxed in a quasi-linear hyperbolic system. The relaxed system is approximated using an upwind scheme converging for fixed relaxation time and vanishing discretization mesh. Further, we use Chapman-Enskog expansion to obtain a viscous scheme for the equilibrium law and prove its convergence to the physical solution. Boundary information is carefully handled in an appropriate inequality linking the entropy numerical flux and the flux function. An extension to general conservative schemes is also investigated.

Keywords—Conservation laws, Stiff source term, Semi-implicit scheme, Relaxing scheme, Relaxed scheme, Boundary condition.

1. INTRODUCTION

Initial boundary value problems for hyperbolic conservation laws play a relevant role in applications. In fact physical phenomena take place in bounded domains under prescribed boundary constraints. Theoretical studies of these problems begin with their well posedness as in [1–3]. The study of the relationship between the initial boundary condition problem for hyperbolic equation and its related viscous parabolic one is investigated in [4] where the authors gave a definition characterizing physical solutions, see [5] for more details. This definition is slightly modified below following [6] where finite volume schemes are treated. A special form to write the boundary condition is exhibited by the vanishing viscosity method. Another formulation of this condition is given by solving Riemann problem. These two formulations are equivalent for linear systems and scalar nonlinear equations as shown in [7] and can be compared in nonlinear systems without being equivalent [7–9]. The measure valued solution case is studied in [10].

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We deal here with a scalar conservation law with stiff source term in the quarter-plan which we can write in the form:

$$u_t + f(u)_x = q(u), \qquad x > 0, \quad t > 0, \tag{1.1}$$

$$u(x,0) = u^0(x), \qquad x \geq 0, \tag{1.2}$$

$$\sup\{\operatorname{sgn}(u(0,t) - u_b(t))(f(u(0,t)) - f(c))\} = 0, \qquad t \geq 0, \tag{1.3}$$

where sgn (u) is the sign of u and where the sup is taken over c lying between $u(0,t)$ and $u_b(t)$. We recall that (1.3) means that $u(0,t) = u_b(t)$ whenever the flow is incoming, i.e., $f'(u(0,t)) > 0$ and that $u_b(t)$ is the desired boundary value.

We will look at (1.1)–(1.3) as an equilibrium for the following linear strictly hyperbolic relaxation system:

$$u_t + v_x = q(u), \qquad x > 0, \quad t > 0, \tag{1.4}$$

$$v_t + a^2 u_x = \frac{1}{\epsilon}(f(u) - v), \qquad x > 0, \quad t > 0, \tag{1.5}$$

$$u(x,0) = u^0(x), \quad v(x,0) = f\left(u^0(x)\right), \qquad x \geq 0, \tag{1.6}$$

$$u(0,t) = u_b(t), \qquad t \geq 0, \tag{1.7}$$

where we assigned only one condition on u at $x = 0$ because there is only one positive characteristic value $a > 0$. We note that the Cauchy problem (1.4)–(1.6) was introduced for the first time by Jin and Xin in [11]. The passage from the relaxation problem (1.4)–(1.7) to the equilibrium one (1.1)–(1.3) may be reached using the expansion of Chapman-Enskog. This leads to the so-called subcharacteristic condition or dissipation condition. In fact, let (u^ϵ, v^ϵ) be the solution of system (1.4)–(1.7). If we consider a formal expansion of Chapman-Enskog type, we get

$$v^\epsilon = f(u^\epsilon) + \epsilon v_1^\epsilon + \epsilon^2 v_2^\epsilon + \cdots.$$

We obtain as a first-order approximation of system (1.4)–(1.7)

$$u_t + f(u)_x = q(u) + \epsilon\left(f'(u)q(u)\right)_x + \epsilon\left(\left(a^2 - f'(u)^2\right)u_x\right)_x. \tag{1.8}$$

Thus, (1.8) is dissipative if the following subcharacteristic condition is satisfied:

$$-a \leq f'(u) \leq a, \qquad \text{for all } u.$$

This condition is essential for the theoretical and numerical studies of the relaxation problems.

Some recent mathematical studies of the relaxation phenomena in absence of the boundary conditions can be found in [11–19] and references therein. It is clear that the main idea in the relaxation theory is to avoid Riemann solvers when constructing numerical schemes for nonlinear conservation laws. It seems that the subcharacteristic stability condition plays a key role and is essential to obtain zero relaxation limit as that the CFL condition is for convergence in numerical schemes.

For relaxation problems with boundary conditions, there is a complete analysis of the small relaxation limit to the equilibrium scalar conservation laws in one space dimension in [20], especially when the initial boundary conditions are some small perturbations of a constant nontransonic state. In [21], more specified stability condition is made on the relaxing system and supplementary proper boundary condition is imposed on the equilibrium system to allow the zero relaxation limit in several dimensions. A convergence result can be found in [22] for a simplified model of diphasic propagation arising in chemical engineering. The long-time effect of the relaxation on the boundary layer is studied in [23,24].

In this work, we study an upwind scheme approaching the unique solution of problem (1.4)–(1.7) when $q = 0$. We again use Chapman-Enskog expansion to deduce, as ϵ goes to zero, a viscous scheme for problem (1.1)–(1.3), we prove its convergence to the unique entropy solution when $u_b = 0$. Thus, we assert that one can approximate the solution u of (1.1)–(1.3) in two ways: tending first h to zero in (3.15)–(3.17) to approximate u^ϵ, solution of (1.4)–(1.7), which is close to u via [20], or tending first ϵ to zero in (3.15)–(3.17) to have the scheme (3.18) approximating u via our Theorem 5.3. We also extend our last result and prove convergence of more general schemes in conservation form for initial boundary value problems.

For the numerical schemes for the initial boundary value problem we can quote [5,25,26], for Godunov scheme, [6] for a finite volume scheme and [5] for Lax-Friedrichs scheme.

This paper is organized as follows: the next section is devoted to recall some preliminaries related to the initial boundary value problems associated with hyperbolic conservation laws including source terms. In Section 3, we present the construction of relaxing and equilibrium schemes. Section 4 supposes $q = 0$ and concerns the convergence of the upwind semi-implicit relaxing scheme. Section 5 supposes $u_b = 0$ and is devoted to the study of the convergence of the equilibrium (relaxed) scheme. Some techniques of this section are exposed in [13,27] but boundary condition imposes careful treatment of the calculus near the boundary. The result of Section 5 is generalized in Section 6 for general TVD schemes satisfying the appropriate condition (6.3) and destined to approximate initial boundary value problems. Concluding remarks are given in Section 7.

2. PRELIMINARIES

We assume that f and q are smooth enough in u and that u^0 and u_b are bounded with bounded total variation. By $u(0,t)$, we denote the trace of u at $x = 0$, i.e., $u(0,t) = \lim_{x\downarrow 0} u(x,t)$. We also have $u(x,0) = \lim_{t\downarrow 0} u(x,t)$. Recall that any u of bounded total variation on $R^+ \times R^+$ has a unique trace at $x = 0$ and at $t = 0$.

In light of [4,6], we define the solution of problem (1.1)–(1.3) as follows.

DEFINITION 2.1. *A function $u \in \mathrm{BV}(R^+ \times R^+)$ is a solution of (1.1)–(1.3) if for all $c \in R$ and all nonnegative test function $\phi \in C_c^1(R^+ \times R^+)$ we have*

$$\begin{aligned}\int_{R^+\times R^+} |u-c|\phi_t\,dx\,dt + \int_{R^+\times R^+} \operatorname{sgn}(u-c)(f(u)-f(c))\phi_x\,dx\,dt \\ + \int_{R^+} \left|u^0(x)-c\right|\phi(x,0)\,dx + \int_{R^+} \operatorname{sgn}(u_b(t)-c)(f(u(0,t)-f(c))\phi(0,t)\,dt \\ + \int_{R^+\times R^+} \operatorname{sgn}(u-c)q(u)\phi\,dx\,dt \ge 0.\end{aligned}$$

Combining the result of Bardos-Leroux-Nedelec [4] and the techniques in the proof of [6, Theorem 2.4] one can easily establish the following theorem.

THEOREM 2.2. *The initial boundary value problem (1.1)–(1.3) admits a unique solution described in Definition 2.1.*

Concerning the relaxation system (1.4)–(1.7), we have the following.

THEOREM 2.3. *If $\|u^0\|_{\mathrm{BV}} + \|u_b\|_{\mathrm{BV}} < \infty$, and if a is large enough, then (1.4)–(1.7) admits a unique solution (u^ϵ, v^ϵ).*

PROOF. Diagonalize the system or see [20].

In addition to the smoothness of f and q, we will suppose

$$|f'| \le a, \qquad \text{(subcharacteristic condition)}, \tag{2.1}$$

$$q(0) = 0, \qquad q' \le 0. \tag{2.2}$$

Conditions (2.1) and (2.2) guarantee, with a CFL condition, the convergence of our schemes described in the next section.

3. RELAXING AND EQUILIBRIUM SCHEMES

In all the sequel, we deal with integer indexes $j \geq 0$ and $n \geq 0$. But if a formula uses $j-1$ or $j-1/2$ (respectively, $n-1$), then it is assumed that $j \geq 1$ (respectively, $n \geq 1$). Let h be the spatial and k be the time grid size and let $\lambda = k/h$ be a fixed positive number. We will use the intervals

$$\begin{aligned} I_0 &= \left[0, \frac{1}{2h}\right[, \\ I_j &= \left[\left(j-\frac{1}{2}\right)h, \left(j+\frac{1}{2}\right)h\right[, \\ I^n &= [nk, (n+1)k[. \end{aligned}$$

For a given family (u_j^n), we define a piecewise function u_h on the quarter plan by

$$u_h(x,t) = u_j^n, \qquad \text{if } (x,t) \in I_j \times I^n.$$

Given this piecewise function u_h, we define on R^+

$$u_h^n(x) = u_j^n, \qquad \text{if } x \in I_j.$$

Other piecewise functions will be defined in the same way $(v_h, w_h, z_h, v_h^n \dots)$. Composed functions will be also naturally denoted as follows:

$$f_j^n = f\left(u_j^n\right), \qquad Q_j^n = Q\left(w_j^n, z_j^n\right), \qquad \dots.$$

We will also use the operator Δ defined by

$$\Delta u_j^n = u_j^n - u_{j-1}^n.$$

Naturally, the initial and the boundary data are projected into the space of piecewise constant functions by

$$\begin{aligned} u_0^0 &= \frac{2}{h} \int_{I_0} u^0(x)\,dx, \\ u_j^0 &= \frac{1}{h} \int_{I_j} u^0(x)\,dx, \qquad j \geq 1, \\ v_j^0 &= f\left(u_j^0\right), \\ u_0^n &= \frac{1}{k} \int_{I^n} u_b(t)\,dt, \qquad n \geq 1. \end{aligned}$$

In order to discretize the relaxation problem (1.4)–(1.7), we first diagonalize it. Let

$$w = au + v \qquad \text{and} \qquad z = au - v$$

be the Riemann invariants corresponding, respectively, to the characteristics $\pm a$. Then w and z verify

$$w_t + aw_x = aq\left(\frac{w+z}{2a}\right) + \frac{1}{\epsilon} Q(w,z), \qquad x > 0, \quad t > 0, \tag{3.1}$$

$$z_t - az_x = aq\left(\frac{w+z}{2a}\right) - \frac{1}{\epsilon} Q(w,z), \qquad x > 0, \quad t > 0, \tag{3.2}$$

where

$$Q(w,z) = f\left(\frac{w+z}{2a}\right) - \frac{w-z}{2}.$$

System (3.1),(3.2) is approximated in two steps. We begin with its linear strictly hyperbolic part. Let (w_h^n, z_h^n) be known. We construct $(w_h^{n+1/2}, z_h^{n+1/2})$ as an approximation at t_{n+1} of the system

$$w_t + aw_x = 0, \qquad x > 0, \quad t > 0, \tag{3.3}$$

$$z_t - az_x = 0, \qquad x > 0, \quad t > 0, \tag{3.4}$$

$$w(x,t_n) = w_h^n(x), \qquad x \geq 0, \tag{3.5}$$

$$z(x,t_n) = z_h^n(x), \qquad x \geq 0, \tag{3.6}$$

$$u(0,t) = u_0^n, \qquad t_n \leq t < t_{n+1}. \tag{3.7}$$

The explicit upwind scheme for this homogeneous system can be written as

$$w_j^{n+1/2} = w_j^n - a\lambda\Delta w_j^n, \tag{3.8}$$

$$z_j^{n+1/2} = z_j^n + a\lambda\Delta z_{j+1}^n. \tag{3.9}$$

REMARK 3.1.

(i) System (3.3)–(3.7) is well posed [1] and scheme (3.8),(3.9) is well defined. In fact, according to (3.4), z is constant throughout the characteristic field $\frac{dx}{dt} = -a$. Thus, knowing $z(x,t)$ and $u(0,t)$, one can deduce

$$z(0,t) = z\left(a\left(t - t^n\right), t^n\right) \qquad \text{and} \qquad w(0,t) = 2au_b(t) - z(0,t).$$

Thus,

$$w_0^{n+1/2} = 2au_0^{n+1} - z_0^{n+1/2}.$$

(ii) We can write, at $(0, t_{n+1})$,

$$z(0,t_{n+1}) = z(ak,t_n) = z(0,t_n) + akz_x(0,t_n) + O\left(k^2\right).$$

So

$$z_0^{n+1/2} = z_0^n + a\lambda\left(z_1^n - z_0^n\right),$$

that is, (3.9) when $j = 0$.

The nonlinear part of (3.1),(3.2) is handled by approximation of an ordinary differential system. Thereby, (w_h^{n+1}, z_h^{n+1}) is an implicit approximation at t_{n+1} of the system

$$\omega' = aq\left(\frac{\omega+\zeta}{2a}\right) + \frac{1}{\epsilon}\,Q(\omega,\zeta),$$

$$\zeta' = aq\left(\frac{\omega+\zeta}{2a}\right) - \frac{1}{\epsilon}\,Q(\omega,\zeta),$$

$$\omega(t_n) = w_j^{n+1/2}, \qquad \zeta(t_n) = z_j^{n+1/2}.$$

Then

$$\frac{\omega(t_{n+1}) - \omega(t_n)}{k} = aq\left(\frac{\omega(t_{n+1}) + \zeta(t_{n+1})}{2a}\right) + \frac{1}{\epsilon}\,Q(\omega(t_{n+1}),\zeta(t_{n+1})),$$

$$\frac{\zeta(t_{n+1}) - \zeta(t_n)}{k} = aq\left(\frac{\omega(t_{n+1}) + \zeta(t_{n+1})}{2a}\right) - \frac{1}{\epsilon}\,Q(\omega(t_{n+1}),\zeta(t_{n+1})),$$

which gives, for w and z,

$$w_j^{n+1} = w_j^{n+1/2} + akq_j^{n+1} + \frac{k}{\epsilon} Q_j^{n+1}, \tag{3.10}$$

$$z_j^{n+1} = z_j^{n+1/2} + akq_j^{n+1} - \frac{k}{\epsilon} Q_j^{n+1}. \tag{3.11}$$

We inject (3.8),(3.9) in (3.10),(3.11) and use Remark 1 for $j = 0$ to obtain:

$$w_0^{n+1} = 2au_0^{n+1} - z_0^n - \lambda a \Delta z_1^n + akq_0^{n+1} + \frac{k}{\epsilon} Q_0^{n+1}, \tag{3.12}$$

$$w_j^{n+1} = w_j^n - \lambda a \Delta w_j^n + akq_j^{n+1} + \frac{k}{\epsilon} Q_j^{n+1}, \tag{3.13}$$

$$z_j^{n+1} = z_j^n + \lambda a \Delta z_{j+1}^n + akq_j^{n+1} - \frac{k}{\epsilon} Q_j^{n+1}. \tag{3.14}$$

In terms of u and v, we can write without difficulty

$$u_j^{n+1} = u_j^n - \frac{\lambda}{2}\left(v_{j+1}^n - v_{j-1}^n\right) + \frac{\lambda a}{2}\left(u_{j-1}^n - 2u_j^n + u_{j+1}^n\right) + kq_j^{n+1}, \tag{3.15}$$

$$v_j^{n+1} = v_j^n - \frac{\lambda a^2}{2}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{\lambda a}{2}\left(v_{j-1}^n - 2v_j^n + v_{j+1}^n\right) + \frac{k}{\epsilon}\left(f_j^{n+1} - v_j^{n+1}\right), \tag{3.16}$$

$$v_0^{n+1} = a\left(u_0^{n+1} - u_0^n\right) + v_0^n - a^2\lambda\Delta u_1^n + a\lambda\Delta v_1^n + \frac{k}{\epsilon}\left(f_0^{n+1} - v_0^{n+1}\right). \tag{3.17}$$

PROPOSITION 3.1. *(u_j^{n+1}, v_j^{n+1}) is uniquely determined by (3.15)–(3.17) when (u_j^n, v_j^n) is known.*

PROOF. The function $u \to u - kq(u)$ is increasing by (2.2).

Concerning the equilibrium equation, we use a formal expansion of Hilbert type in (3.15)–(3.17) to get the following semi-implicit relaxed scheme approximating (1.1)–(1.3):

$$\begin{aligned} v_j^n &= f\left(u_j^n\right), \\ u_j^{n+1} &= u_j^n - \frac{\lambda}{2}\left(f_{j+1}^n - f_{j-1}^n\right) + \frac{\lambda a}{2}\left(u_{j-1}^n - 2u_j^n + u_{j+1}^n\right) + kq_j^{n+1}. \end{aligned} \tag{3.18}$$

REMARK 3.2. We recover the Lax-Friedrichs scheme only if $\lambda a = 1$ in (3.18).

The rest of this paper is devoted to the studies of our schemes described in (3.15)–(3.17) when $q = 0$ and in (3.18) when $u_b = 0$.

4. CONVERGENCE OF THE RELAXING SCHEME

We begin this section by some L^∞ and BV estimates. Obviously, to bound (u, v) variables is equivalent to to bound (w, z) ones. So, for convenience, we will often deal with the Riemann invariants (w, z). Using some techniques from [15,19], we suppose throughout this section that $q = 0$ to recover good monotony properties in the second member of (3.1),(3.2).

PROPOSITION 4.1. L^∞ ESTIMATES. *Assume $q = 0$, the stability condition (2.1) and the CFL condition (4.1):*

$$a\lambda \le 1. \tag{4.1}$$

Suppose, furthermore, the existence of c_1 and c_2 such that the initial and the boundary values satisfy

$$\begin{gathered} c_1 \le u^0(x) \le c_2, \\ a(c_1 + c_2) + f(c_1) - f(c_2) \le 2au_b \le a(c_1 + c_2) + f(c_2) - f(c_1). \end{gathered}$$

Then

$$\left(w_j^n, z_j^n\right) \in \left[c^+, C^+\right] \times \left[c^-, C^-\right],$$

where

$$c^\pm = ac_1 \pm f(c_1), \qquad C^\pm = ac_2 \pm f(c_2).$$

PROOF. The conclusion is obvious for $n = 0$ because $u \to au \pm f(u)$ is nondecreasing. Suppose it is true for some n. We obtain, by (3.13), (3.14), (4.1), and $q = 0$,

$$\begin{aligned} c^+ &\le w_j^{n+1} - \frac{k}{\epsilon} Q_j^{n+1} \le C^+, \qquad j \ge 1, \\ c^- &\le z_j^{n+1} + \frac{k}{\epsilon} Q_j^{n+1} \le C^-. \end{aligned}$$

So, using the monotony of Q, one can prove by absurdity that

$$\begin{aligned} c^+ &\le w_j^{n+1} \le C^+, \qquad j \ge 1, \\ c^- &\le z_j^{n+1} \le C^-. \end{aligned}$$

It remains to justify

$$w_0^{n+1} \in \left[c^+, C^+\right],$$

which follows easily from

$$c^- \le z_0^{n+1} \le C^-$$

and our hypothesis

$$c^+ + C^- \le 2au_b^{n+1} \le c^- + C^+.$$

Now we turn to show that our scheme has a bounded total variation. We need a preliminary result on Q.

LEMMA 4.2. *For all (w, z) and (ω, ζ) in R^2, there exist $0 \le \mu_1, \mu_2 \le 1$ such that $\mu_1 + \mu_2 = 1$ and*

$$Q(w, z) - Q(\omega, \zeta) = -\mu_1(w - \omega) + \mu_2(z - \zeta).$$

PROOF. Use Taylor development, $Q_w \le 0$, $Q_z \ge 0$, and $Q_z - Q_w = 1$.

PROPOSITION 4.3. BV ESTIMATION. *If $\mathrm{TV}(u^0) + TV(u_b) < \infty$, then we have:*

$$\mathrm{TV}\left(w_h^n\right) + \mathrm{TV}\left(z_h^n\right) \le \mathrm{TV}\left(w_h^0\right) + \mathrm{TV}\left(z_h^0\right) + 2a\mathrm{TV}(u_b). \tag{4.2}$$

That is, scheme (3.12)–(3.14) is of bounded x-total variation.

PROOF. Note that $\mathrm{TV}(w_h^0) + \mathrm{TV}(z_h^0)$ is also finite. Next, by Lemma 4.2 and by (3.12)–(3.14), there exist $0 \le \mu_{1j}^n, \mu_{2j}^n \le 1$ such that

$$\begin{aligned} \left(1 + \mu_{11}^n \frac{k}{\epsilon}\right) \left|\Delta w_1^{n+1}\right| &\le (1 - a\lambda) \left|\Delta w_1^n\right| + a\lambda \left|\Delta z_1^n\right| + \mu_{21}^n \frac{k}{\epsilon} \left|\Delta z_1^{n+1}\right| + 2a \left|u_0^{n+1} - u_0^n\right|, \\ \left(1 + \mu_{1j}^n \frac{k}{\epsilon}\right) \left|\Delta w_j^{n+1}\right| &\le (1 - a\lambda) \left|\Delta w_j^n\right| + a\lambda \left|\Delta w_{j-1}^n\right| + \mu_{2j}^n \frac{k}{\epsilon} \left|\Delta z_j^{n+1}\right|, \\ \left(1 + \mu_{2j}^n \frac{k}{\epsilon}\right) \left|\Delta z_j^{n+1}\right| &\le (1 - a\lambda) \left|\Delta z_j^n\right| + a\lambda \left|\Delta z_{j+1}^n\right| + \mu_{1j}^n \frac{k}{\epsilon} \left|\Delta w_j^{n+1}\right|. \end{aligned}$$

We add up the inequalities to get

$$\sum_{j=1}^{J} \left(\left|\Delta w_j^{n+1}\right| + \left|\Delta z_j^{n+1}\right|\right) \le \sum_{j=1}^{J} \left(\left|\Delta w_j^n\right| + \left|\Delta z_j^n\right|\right) + a\lambda \left|\Delta z_{J+1}^n\right| - a\lambda \left|\Delta w_J^n\right| + 2a \left|u_0^{n+1} - u_0^n\right|.$$

Inequality (4.2) is evident for $n = 0$. Assume it is true for some n, so we can go to the limit on J to get

$$\begin{aligned}\mathrm{TV}\left(w_h^{n+1}\right) + \mathrm{TV}\left(z_h^{n+1}\right) &= \mathrm{TV}\left(w_h^{n}\right) + \mathrm{TV}\left(z_h^{n}\right) + 2a\left|u_0^{n+1} - u_0^n\right| \\ &\;\vdots \\ &\leq \mathrm{TV}\left(w_h^{0}\right) + \mathrm{TV}\left(z_h^{0}\right) + 2a\mathrm{TV}(u_b).\end{aligned}$$

To control the global total variations in (x,t) of (w,z) we need to control the distance between v_j^n and $f(u_j^n)$ as follows.

LEMMA 4.4. *If* $\mathrm{TV}(u^0) + \mathrm{TV}(u_b) < +\infty$ *and if* $a\lambda \leq 1$, *then*

$$\frac{k}{\epsilon}\sum_{j\geq 1}\left|Q_j^n\right| \leq \mathrm{TV}\left(w_h^0\right) + \mathrm{TV}\left(z_h^0\right) + 2a\mathrm{TV}\left(u^0\right).$$

PROOF. By Lemma 4.2, there exist μ_{1j}^n and μ_{2j}^n such that

$$Q_j^{n+1} - Q_j^n = -\mu_{1j}^n\left(w_j^{n+1} - w_j^n\right) + \mu_{2j}^n\left(z_j^{n+1} - z_j^n\right).$$

We use scheme (3.12)–(3.14) and $\mu_{1j}^n + \mu_{2j}^n = 1$ to get

$$Q_j^{n+1} - Q_j^n = \mu_{1j}^n a\lambda\Delta w_j^n + \mu_{2j}^n a\lambda\Delta z_{j+1}^n - \frac{k}{\epsilon}Q_j^{n+1},$$

so

$$\left(1 + \frac{k}{\epsilon}\right)\left|Q_j^{n+1}\right| \leq \left|Q_j^n\right| + \left|\Delta w_j^n\right| + \left|\Delta z_{j+1}^n\right|.$$

We denote $r = \epsilon/(\epsilon + k)$ and repeat this inequality to write

$$\left|Q_j^{n+1}\right| \leq r^{n+1}\left|Q_j^0\right| + r^{n+1}\left(\left|\Delta w_j^0\right| + \left|\Delta z_{j+1}^0\right|\right) + \cdots + r\left(\left|\Delta w_j^n\right| + \left|\Delta z_{j+1}^n\right|\right).$$

But $Q_j^0 = Q(w_j^0, z_j^0) = (1/\epsilon)\left(f(u_j^0) - v_j^0\right) = 0$. We sum up on j and we use Proposition 4.3 together with $r^{n+1} + \cdots + r \leq \epsilon/k$ to conclude our proof.

PROPOSITION 4.5. *Under the hypothesis of Propositions 4.1 and 4.3, we have*

$$\sum_{j\geq 0}\left(\left|w_j^{n+1} - w_j^n\right| + \left|z_j^{n+1} - z_j^n\right|\right) \leq c,$$

where c *is a positive constant nondepending on* n *nor on* j.

PROOF. We deduce from (3.12)–(3.14) and (4.1)

$$\sum_{j\geq 1}\left(\left|w_j^{n+1} - w_j^n\right| + \left|z_j^{n+1} - z_j^n\right|\right) \leq \sum_{j\geq 1}\left(\left|\Delta w_j^n\right| + \left|\Delta z_{j+1}^n\right|\right) + 2\,\frac{k}{\epsilon}\sum_{j\geq 1}\left|Q_j^{n+1}\right|.$$

Thus, by Lemma 4.4 and Proposition 4.3,

$$\sum_{j\geq 1}\left(\left|w_j^{n+1} - w_j^n\right| + \left|z_j^{n+1} - z_j^n\right|\right) \leq 3\left(\mathrm{TV}\left(w_h^0\right) + \mathrm{TV}\left(z_h^0\right) + 2a\mathrm{TV}(u_b)\right).$$

but, for $j = 0$,

$$\left|w_0^{n+1} - w_0^n\right| + \left|z_0^{n+1} - z_0^n\right| \leq 4\sup\left\{\left|c^-\right|, \left|c^+\right|, \left|C^-\right|, \left|C^+\right|\right\}.$$

We summarize the L^∞ and the BV estimates in the following.

COROLLARY 4.6. *The approximations (u_h, v_h) defined in (3.15)–(3.17) are bounded in $L^\infty(R^+ \times R^+) \cap \mathrm{BV}_{\mathrm{loc}}(R^+ \times R^+)$.*

PROOF. Let $T = Nk$. The global total variation of w_h on $R^+ \times [0,T]$ is

$$\mathrm{TV}(w_h) \equiv \sum_{n=0}^{N-1} \sum_{j\geq 0} \left|w_j^{n+1} - w_j^n\right| h + \sum_{n=0}^{N-1} \sum_{j\geq 0} \left|w_{j+1}^n - w_j^n\right| k.$$

We get easily, from Propositions 4.3 and 4.5,

$$\mathrm{TV}(w_h) \leq TM,$$

where M is a fixed positive constant. We treat z in the same way and use Proposition 4.1 to conclude.

Now, using Corollary 4.6, the compact embedding of $L^\infty \cap \mathrm{BV}_{\mathrm{loc}}$ in L^1_{loc}, the well-posedness of the relaxation system (1.4)–(1.7) recalled in Theorem 2.3, and classical arguments (cf. [28]), we claim our first main result in this work.

THEOREM 4.7. *Under assumptions of Theorem 2.3, Proposition 4.1, and Proposition 4.3, the approximations (u_h, v_h) defined in (3.15)–(3.17) converge in $L^1_{\mathrm{loc}}(R^+ \times R^+)$ and always everywhere to the unique solution (u^ϵ, v^ϵ) of (1.4)–(1.7) as $h \to 0$.*

REMARK 4.1. One can combine Theorem 4.7 with Theorem 2.3 of [20] to approximate the solutions of (1.1)–(1.3) by tending first h to zero in the scheme (3.15)–(3.17) to obtain (u^ϵ, v^ϵ) solution of (1.4)–(1.7) and tending ϵ to zero in the second time. Thus, the importance of Theorem 4.7.

In the next section, we reverse the order in Remark 1.

5. CONVERGENCE OF THE EQUILIBRIUM SCHEME

In the sequel, we deal with a null boundary condition $u_b = 0$. Thus, we consider

$$u_t + f(u)_x = q(u), \qquad x > 0, \quad t > 0, \tag{5.1}$$

$$u(x,0) = u^0(x), \qquad x \geq 0, \tag{5.2}$$

$$\sup \{\mathrm{sgn}\,(u(0,t))(f(u(0,t)) - f(c))\} = 0, \qquad t \geq 0, \tag{5.3}$$

where the sup is taken over c lying between $u(0,t)$ and zero.

Let us now consider the approximation of problem (5.1)–(5.2) by relaxed scheme (3.18).

To deal with locally total variation we fix $R > 0$, $T > 0$, $T = Nk$, and $J = E(R/h) =$ the integer part of R/h.

PROPOSITION 5.1. *Suppose (2.1), (2.2), and CFL condition (4.1). Assume further that $u_b = 0$, then*

(i) $|u_h^{n+1}|_\infty \leq |u_h^n|_\infty \leq |u_h^0|_\infty$ *(maximum principle),*
(ii) $\mathrm{TV}(u_h^{n+1}) \leq \mathrm{TV}(u_h^n)$ *(TVD property),*
(iii) $\sum_{j=0}^{J} |u_j^{n+1} - u_j^n| \leq c$, *where c does not depend on n.*

PROOF. Such proof is developed in [27], in the absence of boundary conditions. We use again these techniques for the comfort of the reader and for details concerning the boundary.

(i) From (3.18), we can write, after a direct calculation,

$$u_j^{n+1}\left(1 - kq'\left(\alpha_j^{n+1}\right)\right) = u_j^n(1 - a\lambda) - \frac{\lambda}{2}\left(f'\left(\bar{u}_j^n\right) - a\right) u_{j+1}^n + \frac{\lambda}{2}\left(f'\left(\bar{u}_j^n\right) + a\right) u_{j-1}^n,$$

where α_j^{n+1} is between 0 and u_j^{n+1} and verifies

$$q_j^{n+1} - q(0) = q'\left(\alpha_j^{n+1}\right)\left(u_j^{n+1} - 0\right),$$

and $\bar{u}_j^n$ is between u_{j+1}^n and u_{j-1}^n and verifies

$$f_{j+1}^n - f_{j-1}^n = f'(\bar{u}_j^n)(u_{j+1}^n - u_{j-1}^n).$$

Using the subcharacteristic condition (2.1), the nonincreasing property (2.2) of q, and CFL condition (4.1), we get

$$\left|u_j^{n+1}\right| \le \left|u_j^n\right|(1-a\lambda) - \frac{\lambda}{2}\left(f'\left(\bar{u}_j^n\right) - a\right)\left|u_{j+1}^n\right| + \frac{\lambda}{2}\left(f'\left(\bar{u}_j^n\right) + a\right)\left|u_{j-1}^n\right|.$$

Thus,

$$\left|u_j^{n+1}\right| \le \sup\left\{\left|u_{j-1}^n\right|, \left|u_j^n\right|, \left|u_{j+1}^n\right|\right\}$$

and

$$\left|u_h^{n+1}\right|_\infty \le \left|u_h^n\right|_\infty \le \left|u^0\right|_\infty.$$

(ii) Consider:

$$\begin{aligned} v_j^{n+1} &= u_j^n - \frac{\lambda}{2}\left(f_{j+1}^n - f_{j-1}^n\right) + \frac{a\lambda}{2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) \\ &= u_j^n - \frac{\lambda}{2}\Delta f_{j+1}^n - \frac{\lambda}{2}\Delta f_j^n + \frac{a\lambda}{2}\Delta u_{j+1}^n - \frac{a\lambda}{2}\Delta u_j^n. \end{aligned} \tag{5.4}$$

Let $u_{j+1/2}^n$ be such that

$$\Delta f_{j+1}^n = f'\left(u_{j+1/2}^n\right)\Delta u_{j+1}^n.$$

Then

$$v_j^{n+1} = u_j^n - \frac{\lambda}{2}\left(f'\left(u_{j+1/2}\right) - a\right)\Delta u_{j+1}^n - \frac{\lambda}{2}\left(f'\left(u_{j-1/2}\right) + a\right)\Delta u_j^n.$$

So we make use again of condition (2.1) and CFL condition (4.1) to obtain

$$\left|\Delta v_j^{n+1}\right| \le (1-a\lambda)\left|\Delta u_j^n\right| - \frac{\lambda}{2}\left(f'\left(u_{j+1/2}^n\right) - a\right)\left|\Delta u_{j+1}\right| + \frac{\lambda}{2}\left(f'\left(u_{j-3/2}^n\right) + a\right)\left|\Delta u_{j-1}^n\right|.$$

Summing up on j, it becomes

$$\sum_{j\ge 2}\left|\Delta v_j^{n+1}\right| \le \sum_{j\ge 2}\left|\Delta u_j^n\right| + \frac{\lambda}{2}\left(f'\left(u_{3/2}^n\right) - a\right)\left|\Delta u_2^n\right| + \frac{\lambda}{2}\left(f'\left(u_{1/2}^n\right) + a\right)\left|\Delta u_1^n\right|.$$

Using again (3.18) to obtain for some α_j^n between u_{j-1}^{n+1} and u_j^{n+1}

$$\Delta u_j^{n+1}(1 - kq'(\alpha_j^n)) = \Delta v_j^{n+1},$$

which gives, by the nonpositivity of q',

$$\sum_{j\ge 2}\left|\Delta u_j^{n+1}\right| \le \sum_{j\ge 2}\left|\Delta u_j^n\right| + \frac{\lambda}{2}\left(f'\left(u_{3/2}^n\right) - a\right)\left|\Delta u_2^n\right| + \frac{\lambda}{2}\left(f'\left(u_{1/2}^n\right) + a\right)\left|\Delta u_1^n\right|. \tag{5.5}$$

On the boundary, we have $u_0^n = 0$. So, from (3.18),

$$u_1^{n+1} - kq\left(u_1^{n+1}\right) = u_1^n(1-a\lambda) - \frac{\lambda}{2}\Delta f_2^n - \frac{\lambda}{2}\Delta f_1^n + \frac{a\lambda}{2}\Delta u_2^n + \frac{a\lambda}{2}\Delta u_1^n,$$

which can be written as

$$\Delta u_1^{n+1}\left(1-kq'\left(\alpha_j^n\right)\right)=\Delta u_1^n(1-a\lambda)-\frac{\lambda}{2}\left(f'\left(u_{3/2}^n\right)-a\right)\Delta u_2^n-\frac{\lambda}{2}\left(f'\left(u_{1/2}^n\right)-a\right)\Delta u_1^n.$$

Using again $q'\leq 0$, the subcharacteristic condition and the CFL condition, we get

$$\left|\Delta u_1^{n+1}\right|\leq\left|\Delta u_1^n\right|(1-a\lambda)-\frac{\lambda}{2}\left(f'\left(u_{3/2}^n\right)-a\right)\left|\Delta u_2^n\right|-\frac{\lambda}{2}\left(f'\left(u_{1/2}^n\right)-a\right)\left|\Delta u_1^n\right|. \tag{5.6}$$

We conclude (ii) by adding up (5.5) and (5.6).

(iii) Rewrite (3.18) as

$$u_j^{n+1}-u_j^n=-\frac{\lambda}{2}\Delta f_{j+1}^n-\frac{\lambda}{2}\Delta f_j^n+\frac{a\lambda}{2}\Delta u_{j+1}^n-\frac{a\lambda}{2}\Delta u_j^n+kq\left(u_j^{n+1}\right).$$

Use again

$$\Delta f_{j+1}^n=f'\left(u_{j+1/2}^n\right)\Delta u_{j+1}^n,$$

the subcharacteristic condition and the CFL condition to obtain

$$\left|u_j^{n+1}-u_j^n\right|\leq\left|\Delta u_{j+1}^n\right|+\left|\Delta u_j^n\right|+kM,$$

where $M=\sup_{n,j\geq 0}|q(u_j^n)|<\infty$ by (i). Hence,

$$\begin{aligned}\sum_{j=0}^{J}\left|u_j^{n+1}-u_j^n\right|=\sum_{j=1}^{J}\left|u_j^{n+1}-u_j^n\right|&\leq 2\mathrm{TV}\left(u_h^n\right)+Jh\lambda M\\&\leq 2\mathrm{TV}\left(u_h^0\right)+R\lambda M.\end{aligned}$$

COROLLARY 5.2. *The approximations (u_h) defined by (3.18) are bounded in $L^\infty(R^+\times R^+)\cap\mathrm{BV}_{\mathrm{loc}}(R^+\times R^+)$.*

PROOF. Similar to that of Corollary 4.6.

We are now able to set our second main result in this paper.

THEOREM 5.3. *Under assumptions of Proposition 5.1, the approximations (u_h) of the relaxed problem (5.1)–(5.3) defined in (5.18) converges in $L^1_{\mathrm{loc}}(R^+\times R^+)$ and almost everywhere to the unique solution u of (5.1)–(5.3) described by Definition 2.1.*

PROOF.

STEP ONE: DISCRETE ENTROPY INEQUALITY. Consider again v_j^{n+1} defined in (5.4). We have

$$v_j^{n+1}=u_j^n-\lambda\left(g\left(u_j^n,u_{j+1}^n\right)-g\left(u_{j-1}^n,u_j^n\right)\right),$$

where

$$g(u,v)=\frac{1}{2}(f(u)+f(v))-\frac{a}{2}(v-u).$$

Then, using a result of [29] (see also [30]), there exists a numerical entropy flux G associated with the entropy $|u-c|$ such that

$$\left|v_j^{n+1}-c\right|-\left|u_j^n-c\right|+\lambda\left(G_{j+1/2}^n-G_{j-1/2}^n\right)\leq 0,$$

where $G_{j+1/2}^n=G(u_j^n,u_{j+1}^n)$. So by easy calculation, we get

$$\left|u_j^{n+1}-c\right|-\left|u_j^n-c\right|+\lambda\left(G_{j+1/2}^n-G_{j-1/2}^n\right)\leq\operatorname{sgn}\left(u_j^{n+1}-c\right)kq_j^{n+1}, \tag{5.7}$$

since $u_j^{n+1} - v_j^{n+1} = kq(u_j^{n+1})$. Note that one can explicit $G(u,v)$ from [29] as follows:

$$\begin{aligned} G(u,v) &= g(u \vee c, v \vee c) - g(u \wedge c, v \wedge c) \\ &= \frac{1}{2}\operatorname{sgn}(u-c)(f(u)-f(c)) + \frac{1}{2}\operatorname{sgn}(v-c)(f(v)-f(c)) - \frac{a}{2}|v-c| + \frac{a}{2}|u-c|. \end{aligned}$$

Recall that G is continuous with $G(u,u) = \operatorname{sgn}(u-c)(f(u)-f(c))$ which is the entropy flux associated with the entropy function $\eta(u) = |u-c|$.

STEP TWO: CONTINUOUS ENTROPY INEQUALITY. From (5.7), we have

$$h\left(\left|u_j^{n+1}-c\right| - \left|u_j^n - c\right|\right) + k\left(G_{j+1/2} - G_{j-1/2}\right) \le \operatorname{sgn}\left(u_j^{n+1}-c\right) hkq\left(u_j^{n+1}\right).$$

Let ϕ be a test function such that $0 \le \phi \in C_c^1(R^+ \times [0,T[)$ hence,

$$\begin{aligned} \sum_{j\ge1}\sum_{n\ge0} h\left(\left|u_j^{n+1}-c\right| - \left|u_j^n-c\right|\right)\phi_j^n + \sum_{j\ge1}\sum_{n\ge0} k\left(G_{j+1/2}-G_{j-1/2}\right)\phi_j^n \\ \le \sum_{j\ge1}\sum_{n\ge0} \operatorname{sgn}\left(u_j^{n+1}-c\right) hkq_j^{n+1}\phi_j^n, \end{aligned}$$

where $\phi_j^n = \phi(jk, nh)$ and where the summations are finite since ϕ is of compact support. We obtain, after discrete integration by parts,

$$\begin{aligned} &-\sum_{j\ge1} h\left\{\sum_{n\ge1}\left|u_j^n-c\right|\left(\phi_j^n - \phi_j^{n-1}\right) + \left|u_j^0-c\right|\phi_j^0\right\} \\ &-\sum_{n\ge0} k\left\{\sum_{j\ge2} G_{j-1/2}^n\left(\phi_j^n - \phi_{j-1}^n\right) + G_{1/2}^n\phi_1^n\right\} \\ &\le \sum_{j\ge1}\sum_{n\ge0}\operatorname{sgn}\left(u_j^{n+1}-c\right) hkq_j^{n+1}\phi_j^n, \end{aligned}$$

that is,

$$\begin{aligned} &-h\sum_{j\ge1}\sum_{n\ge1}\left|u_j^n-c\right|\left(\phi_j^n-\phi_j^{n-1}\right) - k\sum_{n\ge0}\sum_{j\ge2} G_{j-1/2}^n\left(\phi_j^n-\phi_{j-1}^n\right) \\ &-h\sum_{j\ge1}\left|u_j^0-c\right|\phi_j^0 - \sum_{j\ge1}\sum_{n\ge0}\operatorname{sgn}\left(u_j^{n+1}-c\right) hkq_j^{n+1}\phi_j^n \\ &\le k\sum_{n\ge0} G\left(u_0^n, u_1^n\right)\phi(h, nk). \end{aligned} \tag{5.8}$$

Suppose now that (u_h) converges in L^1_{loc} and almost everywhere to some $u \in \text{BV}$. Then, by classical tools as in the proof of Lax-Wendroff theorem, one can show that the left-hand side of (5.8) goes to

$$\begin{aligned} I_1 = &-\int_{R^+\times R^+} |u-c|\phi_t + \operatorname{sgn}(u-c)(f(u)-f(c))\phi_x \, dx\, dt \\ &-\int_{R^+}\left|u^0(x)-c\right|\phi(x,0)\, dx - \int_{R^+\times R^+}\operatorname{sgn}(u-c)q(u)\phi\, dx\, dt, \end{aligned} \tag{5.9}$$

as h goes to zero. The right-hand side of (5.8) contains the main boundary information and it remains to be over estimated. We need the following inequality linking the numerical entropy flux G and the function flux f near the boundary $x=0$:

$$2G\left(u_0^n, u_1^n\right) \le \operatorname{sgn}\left(u_0^n - c\right)\left(f\left(u_0^n\right) - 2f(c) + f\left(u_1^n\right) + au_0^n - au_1^n\right), \tag{5.10}$$

which can be easily established distinguishing different positions of c with respect to u_0^n and u_1^n and using $|f'| \leq a$. By (5.10) and since $u_0^n = 0$, we get

$$\begin{aligned} k\sum_{n\geq 0} G\left(u_0^n, u_1^n\right)\phi(h,nk) \\ &\leq \frac{1}{2}\sum_{n\geq 0} k\,\mathrm{sgn}\,(-c)\left(f(0)-2f(c)+f\left(u_1^n\right)-au_1^n\right)\phi(h,nk) \\ &\leq \frac{1}{2}\sum_{n\geq 0}\int_{t_n}^{t_{n+1}} \mathrm{sgn}\,(-c)(f(0)-2f(c)+f(u_h(h,t))-au_h(h,t))\phi_h(h,t)\,dt \\ &= \frac{1}{2}\int_{R^+} \mathrm{sgn}\,(-c)(f(0)-2f(c)+f(u_h(h,t))-au_h(h,t))\phi_h(h,t)dt. \end{aligned} \tag{5.11}$$

Let us define

$$\xi_h(t) = au_h(h,t) - f(u_h(h,t)) - f(0).$$

ξ_h is obviously bounded in $L^\infty(R^+)$ by Corollary 5.2. Thus, we can suppose that ξ_h converges to some $\xi \in L^\infty(R^+)$ with respect to the weak* topology. Furthermore, $\phi_h(h,t)$ converges to $\phi(0,t)$ strongly in L^2. Then the second term of (5.11) tends to

$$I_2 = \frac{1}{2}\int_{R^+} \mathrm{sgn}\,(c)(2f(c)+\xi(t))\phi(0,t)\,dt.$$

So as h tends to zero, we have for all $c \in R$

$$I_1 \leq I_2. \tag{5.12}$$

To conclude our proof we take special ϕ. Consider a C_c^1 function ρ_δ such that

$$\begin{aligned} &\rho_\delta(0) = 1, \\ &\rho_\delta(x) = 0, && \text{for } x \geq \delta, \\ &\rho_\delta(x) \in [0,1], && \text{for } x \geq 0, \\ &|\rho_\delta'|_\infty \leq \frac{1}{\delta}. \end{aligned}$$

Take $\phi = \rho_\delta(x)\psi(t)$ where $0 \leq \psi \in C_c^1(]0,t[)$ and tend δ to zero to obtain, for all $c \in R$,

$$\int_{R^+} \mathrm{sgn}\,(u(0,t)-c)\left(f(u(0,t)-f(c)\right)\psi(t)\,dt \leq \frac{1}{2}\int_{R^+} \mathrm{sgn}\,(c)(2f(c)+\xi(t))\psi(t)\,dt.$$

That is,

$$\mathrm{sgn}\,(u(0,t)-c)(f(u(0,t)-f(c)) \leq \frac{1}{2}\,\mathrm{sgn}\,(c)(2f(c)+\xi(t)).$$

Taking $c > \sup\{u(0,t),0\}$ and $c < \inf\{u(0,t),0\}$, it yields

$$f(u(0,t)) = -\frac{1}{2}\xi(t).$$

We inject this in (5.12) to obtain the entropy inequality of Definition 2.1.

STEP THREE: CONCLUSION. Using Corollary 5.2 and the compact embedding of $L^\infty \cap \mathrm{BV}_{\mathrm{loc}}$ in L^1_{loc} we extract from (u_h) a subsequence which converges in L^1_{loc} and almost everywhere to some $u \in \mathrm{BV}$. Step Two above asserts that u is an entropy solution of (5.1)–(5.3) and the uniqueness of the entropy satisfying solution given in Theorem 2.2 ensures that all the sequence (u_h) converges towards the unique solution of (5.1)–(5.3) described by Definition 2.1.

6. GENERAL SCHEMES IN CONSERVATION FORM

Consider a three point difference scheme in the following conservation-like form:

$$u_j^{n+1} = v_j^{n+1} + kq\left(u_j^{n+1}\right), \tag{6.1}$$

$$v_j^{n+1} = u_j^n - \lambda\left(g\left(u_j^n, u_{j+1}^n\right) - g\left(u_{j-1}^n, u_j^n\right)\right), \tag{6.2}$$

where g is a consistent Lipschitz numerical flux. Consider again the Kruskov's entropy and a consistent numerical flux G. For example, G can be given as above by

$$G(u,v) = g(u \vee c, v \vee c) - g(u \wedge c, v \wedge c).$$

Scheme (6.1)–(6.2) is destined to approximate our initial boundary value problem (1.1)–(1.3). Let us generalize Theorem 5.3.

THEOREM 6.1. *Suppose that (6.2) is TVD, L^∞ stable, and that it satisfies a discrete entropy inequality with the numerical entropy G. If q satisfies (2.2) and if G, g, and f satisfy*

$$G(u,v) \leq \operatorname{sgn}(u-c)(g(u,v) - f(c)), \tag{6.3}$$

for all u, v, and c in R, then scheme (6.1) converges to the physical solution of (1.1)–(1.3).

PROOF. Trace the proof of Theorem 5.3 using (6.3) instead of (5.10).

REMARK 6.1. It is condition (6.3) which worked in (5.10) and in Godunov and Lax-Friedrichs schemes of [5].

7. CONCLUSION

In this work, we have established the convergence of a semi-implicit scheme (relaxing scheme) for the boundary value semilinear relaxation problem associated with an initial boundary value problem for conservation laws. We have given also a result of convergence of the relaxed scheme for conservation laws with a boundary condition. This scheme is obtained from the relaxing one using the Chapman-Enskog expansion.

This work may be seen as a starting point for the application of relaxation schemes to the real problems in the systems case, for example.

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